Accurate Solutions of the Radiative Transfer Problem via Theory of Connections

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Introduction: Overview

• The *Remote sensing* is the processes of detecting and monitoring an object or an area by measuring its reflected and emitted radiation. It is widely used in the *Planetary Geology* to study surface properties of Planets and Asteroids.

• The *Transport Theory* represents the theoretical underpinning of remote sensing. *Radiative Transfer Equation* (RTE) describes how radiation and matter interact based on the particle description of light.
• To solve the RTE using the recently developed *Theory of Connections* (ToC) [Mortari 2018]

• The focus of this talk is to show the capability of ToC in solving 1D Radiative Transfer Equation with high accuracy
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Radiative Transfer for Remote Sensing

• Solving radiative transfer problems for remote sensing is generally hard and computationally expensive
  – No direct analytical solutions except in very limited cases

• Solutions to radiative transfer problems for remote sensing generally are
  – Semi-analytical
    ▪ High accuracy in limited cases
  – Numerical
    ▪ Hard implementation
ToC approach to solve Linear ODEs

- ToC derives expressions, called *constrained expressions*, with an embedded set of \( n \) linear constraints

\[
y(t) = g(t) + \sum_{k=1}^{n} \eta_k p_k(t) = g(t) + \eta^T p(t)
\]

- According to the literature, the \( g(t) \) used will be an expansion of orthogonal polynomials (Chebyshev)
  - The solution of the problem is reduced to the calculation of the coefficients of the expansion of Chebyshev polynomials

- ToC has been used to solve several kind of problems, both linear and non-linear, in different areas
  - **Energy Optimal Landing Guidance** – linear- [Furfaro and Mortari 2018];
  - **Fuel Efficient Landing Guidance** – non linear- [Schiassi, Furfaro, et. Al 2019]
    - Machine error accuracy in milliseconds
**Application** of the ToC for the solution of a linear ODE, with two constraints at two points:

\[ k_2(t)\ddot{y}(t) + k_1(t)\dot{y}(t) + k_0(t)y(t) = f(t) \]  

subject to:

\[ \begin{align*} 
  y(t_0) &= y_0 \\
  y(t_f) &= y_f 
\end{align*} \]

- Change of independent variable, to be able to use an expansion of orthogonal polynomials from \( t \in [t_0, t_f] \) to \( x \in [x_0, x_f] \), where \( x_0 = -1, \ x_f = 1 \).

The new variable \( x \) is defined as:

\[ x = c(t - t_0) + x_0 \]

Where \( c \) is the integration range ratio:

\[ c = \frac{x_f - x_0}{t_f - t_0} \]

Due to the derivative chain rule, it follows that:

\[ \begin{align*} 
  y(t) &= y(x) \\
  \frac{dy}{dt} &= \dot{y} = c \frac{dy}{dx} = cy' \\
  \frac{d^2y}{dt^2} &= \ddot{y} = c^2 \frac{d^2y}{dx^2} = c^2 y'' 
\end{align*} \]
ToC approach to solve Linear ODEs

• Replacing in the equation we get:

\[ c^2 k_2 y''(x) + c k_1 y'(x) + k_0 y(x) = f(x) \]

subject to:

\[ \begin{cases} y(x_0) = y_0 \\ y(x_f) = y_f \end{cases} \]

• Constrained expressions

\[
\begin{align*}
  y(x) &= g(x) + \eta_1 p(x) + \eta_2 q(x) \\
  y'(x) &= g'(x) + \eta_1 p'(x) + \eta_2 q'(x) \\
  y''(x) &= g''(x) + \eta_1 p''(x) + \eta_2 q''(x) \\
  g(x) &= h^T(x) \xi \\
  g'(x) &= h'^T(x) \xi \\
  g''(x) &= h''^T(x) \xi
\end{align*}
\]
ToC approach to solve Linear ODEs

- Using the boundary conditions, we find $\eta_1, \eta_2$

$$\begin{aligned}
\begin{cases}
y_0 &= g_0 + \eta_1 p_0 + \eta_2 q_0 \\
y_f &= g_f + \eta_1 p_f + \eta_2 q_f
\end{cases} & \Rightarrow 
\begin{bmatrix} p_0 & q_0 \\ p_f & q_f \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} y_0 - g_0 \\ y_f - g_f \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\eta_1 &= \frac{1}{\Delta} \left( q_0 h_f^T - q_f h_0^T \right) \xi + \frac{1}{\Delta} \left( q_f y_0 - q_0 y_f \right) \\
\eta_2 &= \frac{1}{\Delta} \left( p_0 h_f^T + p_f h_0^T \right) \xi + \frac{1}{\Delta} \left( p_0 y_f - p_f y_0 \right)
\end{aligned}$$

where: $\Delta = p_0 q_f - q_0 p_f \neq 0$
By replacing the newly found values of $\eta_1, \eta_2$ in the constrained expressions, we get:

\[
y(x) = \left[ h^T(x) + \frac{p(x)}{\Delta} (q_0 h_f^T - q_f h_0^T) + \frac{q(x)}{\Delta} (q_f h_0^T - p_0 h_f^T) \right] \xi + \left[ \frac{p(x)}{\Delta} (q_f y_0 - q_0 y_f) + \frac{q(x)}{\Delta} (p_0 y_f - p_f y_0) \right]
\]

\[
y'(x) = \left[ h'^T(x) + \frac{p'(x)}{\Delta} (q_0 h_f^T - q_f h_0^T) + \frac{q'(x)}{\Delta} (q_f h_0^T - p_0 h_f^T) \right] \xi + \left[ \frac{p'(x)}{\Delta} (q_f y_0 - q_0 y_f) + \frac{q'(x)}{\Delta} (p_0 y_f - p_f y_0) \right]
\]

\[
y''(x) = \left[ h''^T(x) + \frac{p''(x)}{\Delta} (q_0 h_f^T - q_f h_0^T) + \frac{q''(x)}{\Delta} (q_f h_0^T - p_0 h_f^T) \right] \xi + \left[ \frac{p''(x)}{\Delta} (q_f y_0 - q_0 y_f) + \frac{q''(x)}{\Delta} (p_0 y_f - p_f y_0) \right]
\]

We define the following parameters:

\[
\begin{align*}
    aa &= \left( \frac{q_0 h_f^T - q_f h_0^T}{\Delta} \right) \\
    bb &= \left( \frac{q_f h_0^T - p_0 h_f^T}{\Delta} \right) \\
    cc &= \left( \frac{q_f y_0 - q_0 y_f}{\Delta} \right) \\
    dd &= \left( \frac{p_0 y_f - p_f y_0}{\Delta} \right)
\end{align*}
\]
Then:

\[ y(x) = [h^T(x) + p(x)aa + q(x)bb] \xi + [p(x)cc + q(x)dd] \]

\[ y'(x) = [h'^T(x) + p'(x)aa + q'(x)bb] \xi + [p'(x)cc + q'(x)dd] \]

\[ y''(x) = [h''^T(x) + p''(x)aa + q''(x)bb] \xi + [p''(x)cc + q''(x)dd] \]

• By plugging into:

\[ c^2 k_2 y''(x) + ck_1 y'(x) + k_0 y(x) = f(x) \]

we obtain the equation with the following form:

\[ c^2 k_2 \left\{ [h''^T(x) + p''(x)aa + q''(x)bb] \xi + [p''(x)cc + q''(x)dd] \right\} + 

\[ ck_1 \left\{ [h'^T(x) + p'(x)aa + q'(x)bb] \xi + [p'(x)cc + q'(x)dd] \right\} + 

\[ k_0 \left\{ [h'^T(x) + p'(x)aa + q'(x)bb] \xi + [p'(x)cc + q'(x)dd] \right\} = f(x) \]
By rearranging the terms of the equation just obtained, the solution of the initial ODE is reduced to the solution of a Linear System \( A \xi = b \)

\[
\{A_{ij}\} = (c^2k_2h''_{ij} + ck_1h'_{ij} + k_0h_{ij}) + (c^2k_2p''_i + ck_1p'_i + k_0p_i)aa_j + (c^2k_2q''_i + ck_1q'_i + k_0q_i)bb_j
\]

\[
\{b_i\} = f_i - (c^2k_2p''_i + ck_1p'_i + k_0p_i)cc - (c^2k_2q''_i + ck_1q'_i + k_0q_i)dd
\]

- Once we get \( \xi \) by a Least-Squares, it is replaced in the constrained expressions shown previously:

\[
y(x) = h^T(x)\xi + \eta_1p(x) + \eta_2q(x)
\]

\[
y'(x) = h'^T(x)\xi + \eta_1p'(x) + \eta_2q'(x)
\]

\[
y''(x) = h''^T(x)\xi + \eta_1p''(x) + \eta_2q''(x)
\]
• We can now reconstruct the solution of the initial equation as a function of $t$:

$$y(t) = y(x) \quad \dot{y}(t) = cy'(x) \quad \ddot{y}(t) = c^2 y''(x)$$

• Finally, by calculating the residuals, we can check the precision of the equation:

$$Res = k_2 \dot{y}(t) + k_1 \dot{y}(t) + k_0 y(t) - f(t)$$

**PRECISION OF THE EQUATION** \(\propto\) \(\frac{1}{Res}\)
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Basic Formulation of the Radiative Transfer Problem

The Radiative Transfer Equation RTE (according to Chandrasekhar) to be solved is:

\[
\mu \frac{\partial}{\partial t} I(\tau, \mu, \phi) + I(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I(\tau, \mu', \phi') d\phi' d\mu'
\]

With the following constraints:

\[
\begin{align*}
I(0, \mu, \phi) &= \pi \delta(\mu - \mu_0) \delta(\phi - \phi_0) & \text{for } \mu > 0 \\
I(\Delta, \mu, \phi) &= 0 & \text{for } \mu < 0
\end{align*}
\]
• Separation of the Intensity into uncollided fraction and collided fraction (or diffused)

• Making use of the Addition Theorem of the Spherical Harmonics to express the phase function:

\[ p(\cos \Theta) = \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} \beta_l P_l^m(\mu') P_l^m(\mu) \cos[m(\phi' - \phi)] \]

• Expression of the diffused fraction by Fourier series (Siewert 1998):

\[ I^*(\tau, \mu, \phi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) I_m(\tau, \mu) \cos[m(\phi' - \phi)] \]

where the \( m \)-th Fourier component satisfies the equation of transfer

\[ \mu \frac{\partial}{\partial \tau} I_m(\tau, \mu) + I_m(\tau, \mu) = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I_m(\tau, \mu') d\mu' + \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(\mu_i) \]

• Discretization of \( \mu \rightarrow \mu_i \) where \( i = 1, \ldots, N \)
• Change of variable: from $\tau$ to $x$.
• Splitting of the equation in forward flux and backward flux.
• Gauss-Legendre quadrature for calculating the integral in the range [0,1].

$$c\mu_i \frac{\partial}{\partial x} I_m^+ + I_m^+ = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) \sum_{k=1}^{N} w_k P_l^m(\mu_k) [I_m^+ + (-1)^{l-m} I_m^-] + \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(\mu_i)$$

$$-c\mu_i \frac{\partial}{\partial x} I_m^- + I_m^- = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(-\mu_i) \sum_{k=1}^{N} w_k P_l^m(-\mu_k) [(-1)^{l-m} I_m^+ + I_m^-] + \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(-\mu_i)$$

with following boundary conditions:

$$I_m^+(0) = 0$$

$$I_m^-(\Delta) = 0$$
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Formulation via ToC

- Constrained expressions:

\[ I_m^+(x) = g^+(x) + \eta^+ p(x) \]
\[ I_m^-(x) = g^-(x) + \eta^- q(x) \]

\[ I_0^+ = 0 = g_0 + \eta^+ \]
\[ I_0^- = 0 = g_0 + \eta^- \]

- Use of boundary conditions to find the coefficients \( \eta \):

\[ \eta^+ = -g_0 \]
\[ \eta^- = -g_f \]

- Replacement of \( \eta \) in the constrained expressions:

\[ I_m^+(x) = g^+(x) - g_0 \]
\[ I_m^-(x) = g^-(x) - g_f \]

- Finally, we get the solutions in the following forms:

\[ I_m^+ = (h^T - h_0^T) \cdot \xi^+ \]
\[ I_m^- = (h^T - h_f^T) \cdot \xi^- \]
Solution of the RTE via ToC: Formulation

• Replacement of the constrained expressions in the two DEs

$$(c \mu_i h' + h - h_0) \cdot \xi_i^+ - \frac{\omega}{2} \sum_{l=m}^L \beta_l P_i^m(\mu_i) \sum_{k=1}^N w_k P_1^m(\mu_k) (h - h_0) \cdot \xi_k^+$$

$$- \frac{\omega}{2} \sum_{l=m}^L \beta_l P_1^m(\mu_i) \sum_{k=1}^N w_k P_l^m(\mu_k) (-1)^{l-m} (h - h_f) \cdot \xi_k^- = \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^L \beta_l P_1^m(\mu_0) P_1^m(\mu_i)$$

$$(c \mu_i h' + h - h_0) \cdot \xi_i^- - \frac{\omega}{2} \sum_{l=m}^L \beta_l P_i^m(-\mu_i) \sum_{k=1}^N w_k P_1^m(-\mu_k) (h - h_f) \cdot \xi_k^-$$

$$- \frac{\omega}{2} \sum_{l=m}^L \beta_l P_i^m(-\mu_i) \sum_{k=1}^N w_k P_l^m(-\mu_k) (-1)^{l-m} (h - h_0) \cdot \xi_k^+ = \frac{\omega}{2} e^{-\tau/\mu_0} \sum_{l=m}^L \beta_l P_1^m(\mu_0) P_l^m(-\mu_i)$$

• Computation of the coefficients $\xi$ by solving the system $A \cdot \xi = b$

of dimensions: $(2MN \times 2mN) \cdot (2mN \times 1) = (2MN \times 1)$

$M =$ spatial discretization points  
$N =$ angle discretization points  
$m =$ number of polynomials
Solution of the RTE via ToC: Formulation

where:

\[ \bullet_i = c \mu_i h' + h - h_0 \]

\[ \bullet_i = -c \mu_i h' + h - h_f \]

\[ \star^k_i = -\frac{\omega}{2} w_k (h - h_0) \sum_{l=m}^{L} \beta_l P_l(\mu_i)P_l(\mu_k); \]

\[ \star^k_i = -\frac{\omega}{2} w_k (h - h_f) \sum_{l=m}^{L} \beta_l P_l(\mu_i)P_l(\mu_k) (-1)^{l-m} \]

\[ \star^k_i = -\frac{\omega}{2} w_k (h - h_0) \sum_{l=m}^{L} \beta_l P_l(-\mu_i)P_l(-\mu_k) (-1)^{l-m}; \]

\[ \star^k_i = -\frac{\omega}{2} w_k (h - h_f) \sum_{l=m}^{L} \beta_l P_l(-\mu_i)P_l(-\mu_k) \]
Solution of the RTE via ToC: Formulation

- Substitution of $\xi$ coefficients in the constrained expressions

\[ I^+_m = (h - h_0) \cdot \xi^+ \quad I^-_m = (h - h_f) \cdot \xi^- \]

- Substitution of the $m$-th Fourier series component:

\[ I^+_m(\tau, \mu, \phi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) I^+_m(\tau, \mu) \cos[m(\phi' - \phi)] \quad I^-_m(\tau, \mu, \phi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) I^-_m(\tau, \mu) \cos[m(\phi' - \phi)] \]

- **Post-processing**, to find solutions at every polar angle, and at any slab’s point (via ToC)

\[
(c \gamma_j h' + h - h_0) \cdot \xi^+_j = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(\gamma_j) \sum_{k=1}^{N} w_k P_l^m(\mu_k) [(h - h_0) \cdot \xi^+_k + (-1)^{l-m} (h - h_f) \cdot \xi^-_k] + b^+_j
\]

\[
(-c \gamma_j h' + h - h_f) \cdot \xi^-_j = \frac{\omega}{2} \sum_{l=m}^{L} \beta_l P_l^m(-\gamma_j) \sum_{k=1}^{N} w_k P_l^m(-\mu_k) [(-1)^{l-m} (h - h_0) \cdot \xi^+_k + (h - h_f) \cdot \xi^-_k] + b^-_j
\]

The new arbitrary angles are $\gamma_j$ and the new unknown vector is $\xi^+_j$, computed by a Least-Squares method.
The accuracy of this new method for the RTE solution was validated by comparing the results with the benchmarks published by C.E. Siewert et. al, for the following case studies:

- Isotropic, Two-Stream Approximation;
- Isotropic, Multi-stream;
- Anisotropic, Mie Scattering;
- Anisotropic, Haze L Problem.

For all the cases considered, we obtained the same digits published by Garcia & Siewert (1986)
Solution of the RTE via ToC: Results

Haze L Problem
for $m=0$ Fourier component
(normal incident beam and conservative case)

\[ \omega = 1 \]
\[ \mu_0 = 1 \]
\[ \Delta = 1 \]

**RTE via ToC**

- $N = 30$

**RTE via ADO**

- $N = 100 \div 150$

- CPU time for the Least-Squares $\cong 8.5$ seconds
- Total CPU time to run the code (including matrices construction, plots and post-processing) $\cong 25$ seconds
Solution of the RTE via ToC: Results

Haze L Problem for 83 $m$ Fourier components

\[ \omega = 0.9 \]
\[ \mu_0 = 0.5 \]
\[ \Delta = 1 \]
\[ \phi - \phi_0 = \pi/2 \]

---

**RTE via ToC**

N = 30

**RTE via ADO**

N = 100 ÷ 150

---

- CPU time per Least-Squares for each $m \approx 8.5$ seconds
- Total CPU time for any Least-Squares $\approx 11.7$ minutes
- Total CPU time to run the code (including matrices construction, plots and post-processing) $\approx 25$ minutes

---

### Table 13: Haze L Problem - The Intensity $I_\omega(\tau, \mu, \phi)$ for the Haze L phase function with $\omega = 0.9$, $\mu_0 = 0.5$, and $\phi - \phi_0 = \pi/2$.  

<table>
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<tr>
<th>$\mu$</th>
<th>$\tau = 0$</th>
<th>$\tau = 0.05\Delta$</th>
<th>$\tau = 0.1\Delta$</th>
<th>$\tau = 0.2\Delta$</th>
<th>$\tau = 0.5\Delta$</th>
<th>$\tau = 0.75\Delta$</th>
<th>$\tau = \Delta$</th>
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<td>2.14170e-2</td>
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<td>1.71573e-2</td>
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• Conclusions and Outlooks
Conclusions and Outlooks

• The RTE is solved via the recently developed ToC
  ▪ The accuracy of the results is compared with the recognized benchmarks
  ▪ Straightforward implementation
  ▪ Reformulation not required for the conservative case $\omega=1$

• Future developments
  ▪ To use this new methodology to compute the Reflectance, for the study of asteroid binary systems properties through light-curves inversions
  ▪ To solve RTE for the multi-slab case
  ▪ To solve the 3D time-dependent RTE
Thanks for the attention

Questions time