Physics-Informed Neural Networks Applied to a Series of Constrained Space Guidance Problems

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Introduction

• Optimal Control Problems (OCPs) represent an essential field within space engineering.
  • It is important to design minimum time, fuel/energy optimal trajectories for space missions.

• The goal of this work is to present a new methodology to solve Constrained Optimal Control Problems (COCPs) for space guidance by means of the novel Physics-Informed Neural Network (PINN) framework named Extreme Theory of Functional Connections (X-TFC).
  • Indirect method exploiting the Pontryagin Minimum Principle (PMP) is used to retrieve the optimal control.
  • Problems considered: Feldbaum problem (typical OCP), minimum time – energy optimal Halo - Halo transfer, 1D fuel optimal lunar landing.
Constrained Optimal Control Problems (COCPs)

**Constrained Optimal Control Problem (COCP)**

\[
J = \Phi(x(t_0), t_0, x(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(x(t), u(t), t) \, dt
\]

\[
\dot{x} = f(x(t), u(t), t), \quad \Phi(x(t_0), t_0) = \Phi_0, \quad \Phi(x(t_f), t_f) = \Phi_f
\]

\[u_i \in [d_i^-, d_i^+]\]

**New unconstrained control variable with saturation function**

\[
\phi_i(w_i) = d_i^+ - \frac{d_i^+ - d_i^-}{1 + \exp(sw_i)} \quad \text{with} \quad s = \frac{c}{d_i^+ - d_i^-}
\]

**Equality constraints in the Hamiltonian + first-order necessary conditions**

**Transversality conditions**

\[
\lambda(t_0) = -\frac{\partial J}{\partial x_0}, \quad H(t_0) = \frac{\partial J}{\partial t_0}, \quad \lambda(t_f) = \frac{\partial J}{\partial x_f}, \quad H(t_f) = \frac{\partial J}{\partial t_f}
\]
Physics-Informed Neural Networks

- (Data) + Neural Networks + Physics Laws = Physics-Informed Neural Networks (PINN)
- PINNs are a newly developed framework for solving parametric DEs
  - The physics laws (modeled via parametric DEs), and eventually data, drive the training of the network

\[
f \left( x; \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_d}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_d}; \ldots; \lambda \right) = 0, \quad x \in \Omega, \quad B(u, x) = 0 \quad \text{on} \quad \partial \Omega.
\]

The Theory of Functional Connections (TFC) [Mortari, 2017] is a recently developed framework for functional interpolation

- The functions are approximated via a constrained expression
  - Sum of a free-chosen function and a functional that analytically satisfies the constraints
- TFC can be applied to solve DEs
  - The free-chosen function is an expansion of Chebyshev polynomials
  - The constraints are the Initial/Boundary Conditions (IC or BC)

The Physics-Informed Neural Network (PINN) Methods are a novel approach, coming from the Machine Learning community

- The DE latent solutions are approximated via a (Deep) Neural Network (NN), and the DEs drive the NN training (i.e., it acts as regulator)

### Pros

<table>
<thead>
<tr>
<th>TFC (w/ Cheb. Pol.)</th>
<th>PINN</th>
</tr>
</thead>
<tbody>
<tr>
<td>ICs/BCs always analytically satisfied</td>
<td>Expanding the latent solution via NN allows to apply this method to solve high order PDEs (e.g. no cur)</td>
</tr>
<tr>
<td>Accurate Solutions</td>
<td></td>
</tr>
<tr>
<td>Low Computational Time</td>
<td></td>
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</tbody>
</table>

### Cons

<table>
<thead>
<tr>
<th>TFC (w/ Cheb. Pol.)</th>
<th>PINN</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-linear problems can be very sensitive to the initial guesses</td>
<td>Many training points required for high accuracy (ICs/BCs not analytically satisfied)</td>
</tr>
<tr>
<td>It suffers of the curse of dimensionality when solving ODE/PDE problems</td>
<td>Computational expensive when gradient based methods are used to train the NN</td>
</tr>
</tbody>
</table>
The Physics-Informed Extreme Theory of Functional Connections (X-TFC) is a synergy of the TFC and the standard PINN methods that helps to overcome their limitations for solving DEs.
Extreme Theory of Functional Connections (X-TFC)

- X-TFC uses the TFC constrained expression where the free-chosen function $g$ is a Single Layer Feedforward NN (SLNN) trained via Extreme Learning Machine (ELM) Algorithm [Huang et al., 2006].

**X-TFC approach to solving generic DEs**

1. Approximate the Latent Solution(s) with the CE
2. Analytically Satisfy the ICs/BCs
3. Expand with NN
4. Substitute into the (S)DEs
5. Build the Loss(es)
6. Train the NN
7. Build the Approximate Solution(s)
• The Jacobian matrix required for an eventual iterative least-square procedure can be computed either analytically, or by the symbolic computation, or the automatic differentiation toolbox.
Extreme Learning Machine (ELM)

- ELM is a training algorithm for Single Layer Feedforward Neural Network (SLNN) that randomly selects input weights and bias, and computes the output weights ($\boldsymbol{\beta}$) via least-square.
  
  - Input weights ($w_i$) and bias ($b_i$) are not tuned during the training.

- The convergence of the ELM algorithm is proved by Huang et al (2006).
  
  - The convergence is guaranteed for any input weights and bias randomly chosen from any continuous probability distribution.
The Feldbaum problem is a typical optimal control problem.

Original COCP

\[
\begin{align*}
\min \ J &= \frac{1}{2} \int_0^1 (f^2 + u^2) \, dt \\
\text{subject to} \quad & \dot{f} = \frac{df}{dt} = -f + u \\
& 0 \leq t \leq 1 \\
& f(0) = 1 \\
& u \in [u_{\text{min}}, u_{\text{max}}]
\end{align*}
\]

Modified UOCP

\[
\begin{align*}
\min \ J &= \frac{1}{2} \int_0^1 (f^2 + u^2) \, dt + \epsilon \int_0^1 w^2 \, dt \\
H(t) &= \frac{1}{2} (f^2 + u^2) + \lambda (-f + u) + \epsilon w^2 + \nu (u - \phi(w)) \\
\frac{\partial H}{\partial u} &= u + \lambda + \nu = 0 \\
\frac{\partial H}{\partial w} &= 2\epsilon w - \nu \phi'(w) = 0 \\
\dot{f} &= \frac{\partial H}{\partial \lambda} = -f + u \\
\dot{\lambda} &= -\frac{\partial H}{\partial f} = \lambda - f \\
u &= \phi(w)
\end{align*}
\]

The following transversality condition has to be applied: \( \lambda(1) = \lambda_f = 0 \).
Feldbaum Problem

• The CEs and their derivatives are:

\[
\begin{align*}
f &= (\sigma - \Omega_1 \sigma_0)^T \beta_f + \Omega_1 f_0 \\
\lambda &= (\sigma - \Omega_1 \sigma_f)^T \beta_\lambda + \Omega_1 \lambda_f \\
w &= \sigma^T \beta_w \\
u &= \sigma^T \beta_u \\
\nu &= \sigma^T \beta_\nu
\end{align*}
\]

\[
\begin{align*}
f' &= b^2 \left[ (\sigma' - \Omega'_1 \sigma_0)^T \beta_f + \Omega'_1 f_0 \right] \\
\lambda' &= b^2 \left[ (\sigma' - \Omega'_1 \sigma_f)^T \beta_\lambda + \Omega'_1 \lambda_f \right]
\end{align*}
\]

• \( \Omega \) are the **switching functions** and their analytical expressions are computed by imposing the boundary conditions.

• The unknowns of the problem are: \( \beta = \{ \beta_f \ \beta_\lambda \ \beta_w \ \beta_u \ \beta_\nu \}^T \)

• The losses of the associated Two-Point Boundary Value Problem (TPBVP) are:

\[
\begin{align*}
\mathcal{L}_f &= \dot{f} + f - u \\
\mathcal{L}_\lambda &= \dot{\lambda} - \lambda + f \\
\mathcal{L}_u &= u + \lambda + \nu \\
\mathcal{L}_w &= \epsilon w - \nu \phi'(w) \\
\mathcal{L}_d &= u - \phi(w)
\end{align*}
\]

Mapping coefficient from \( t \) in \([t_0; t_f]\) to \( z \) in \([-1;1]\)

\[
b^2 = c = \frac{z_f - z_0}{t_f - t_0}
\]
Feldbaum Problem

- Parameters employed for the Feldbaum problem:
  
<table>
<thead>
<tr>
<th>$n$</th>
<th>$L$</th>
<th>$\sigma$</th>
<th>$[u_{\text{min}}, u_{\text{max}}]$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>25</td>
<td>Gaussian</td>
<td>[-0.2,0]</td>
<td>4</td>
</tr>
</tbody>
</table>

- For all the chosen problems, the **initial guess** of the unknowns are chosen **randomly** within the interval (0,1).
- In order to make the UOCP close enough to the original COCP, a continuation procedure has been applied to decrease the value of $\epsilon$ from 1 to $10^{-5}$, while maintaining a good accuracy.
- Decreasing the value of $\epsilon$ induces higher values of the fictitious control $w$, which approaches the asymptotic limits.
Feldbaum Problem

- Results for the constrained Feldbaum problem with $\epsilon = 10^{-5}$

Trajectory and control plus the comparison with the unconstrained Feldbaum problem version

Performances of the constrained Feldbaum problem
Minimum time – energy optimal Halo-Halo transfer

• The Circular Restricted Three Body Problem (CR3BP) framework is employed to study constrained minimum time – energy optimal Halo-Halo transfers in the Earth-Moon system (dimensionless units are considered).

• Due to the complexity of the problem, a 2-loop optimization procedure is employed. An outer loop based on the Particle Swarm Optimization (PSO) is used to minimize the time, whereas the fixed time - energy optimal problem is solved rapidly via indirect method and X-TFC in the inner loop.

\[ \min J = \Gamma t_f + \frac{1}{2} \int_{t_0}^{t_f} u^T u \, dt + \epsilon \int_{t_0}^{t_f} w^T w \, dt \quad \text{subject to:} \]

\[
\begin{align*}
\dot{r} &= v \\
\dot{v} &= \nabla U(r) + Mv + u \\
t_0 \leq t \leq t_f \\
r(t_0) &= r_0 \\
v(t_0) &= v_0 \\
r(t_f) &= r_f \\
v(t_f) &= v_f \\
u_{\min} \leq u_i \leq u_{\max}
\end{align*}
\]

• The Hamiltonian of the problem is:

\[ H = \frac{1}{2} u^T u + \epsilon w^T w + \lambda_r^T v + \lambda_v^T \left( \nabla U(r) + Mv + u \right) + \nu^T (u - \phi(w)) \]
Minimum time – energy optimal Halo-Halo transfer

- By using the CEs, the latent solutions are approximated.

\[
\begin{align*}
\dot{r} &= \left(\sigma - \Omega_1 \sigma_0 - \Omega_2 \sigma_f - \Omega_3 \sigma'_0 - \Omega_4 \sigma'_f\right)^T \beta_r + \Omega_1 r_0 + \Omega_2 r_f + \frac{\Omega_3 v_0 + \Omega_4 v_f}{l^2} \\
\dot{\lambda}_r &= \sigma^T \beta_{\lambda_r} \\
\dot{\lambda}_v &= \sigma^T \beta_{\lambda_v} \\
\dot{w} &= \sigma^T \beta_w \\
\dot{\nu} &= \sigma^T \beta_{\nu}
\end{align*}
\]

- The first order necessary conditions and the corresponding losses are:

\[
\begin{align*}
\dot{u} &= \frac{\partial H}{\partial \lambda_r} = v \\
\dot{v} &= \frac{\partial H}{\partial \lambda_v} = \nabla U(r) + Mv + u \\
\dot{\lambda}_r &= \frac{\partial H}{\partial r} = -\nabla \nabla^T \lambda_v \\
\dot{\lambda}_v &= \frac{\partial H}{\partial v} = -\lambda_r - M^T \lambda_v \\
\dot{\nu} &= \frac{\partial H}{\partial \nu} = u - \phi(w) = 0 \\
\mathcal{L}_u &= \dot{v} - \nabla U(r) - Mv - \phi(w) \\
\mathcal{L}_{\lambda_r} &= \dot{\lambda}_r + \nabla \nabla^T \lambda_v \\
\mathcal{L}_{\lambda_v} &= \dot{\lambda}_v + \lambda_r + M^T \lambda_v \\
\mathcal{L}_{\nu} &= \phi(w) + \lambda_v + \nu \\
\mathcal{L}_w &= 2\epsilon w - \nu \odot \phi'(w)
\end{align*}
\]
Minimum time – energy optimal Halo - Halo transfer

- The goal is to pass from a L1 Halo to a L2 Halo orbit.
- For PSO, the cost function associated to the particles is the sum of the time of flight and the norm of the loss vector.
- The other parameters used for this example are:

<table>
<thead>
<tr>
<th>n</th>
<th>L</th>
<th>σ</th>
<th>([u_{i,\text{min}}, u_{i,\text{max}}]) km/s^2</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>35</td>
<td>Logistic</td>
<td>([-5, 5]) \cdot 10^{-7}</td>
<td>10</td>
</tr>
</tbody>
</table>

- A continuation procedure has been applied to decrease the value of \(\epsilon\) from 1 to \(10^{-6}\).
- The following boundary conditions are considered:

  \[ r_0 = [0.823385182067467, 0, -0.02277562730235] \]
  \[ v_0 = [0, 0.134184170262437, 0] \]
  \[ r_f = [1.119601641146371, 0, 0.010405691913477] \]
  \[ v_f = [0, 0.178317957880263, 0] \]
Minimum time – energy optimal Halo - Halo transfer

- Time of flight = 9.98 days.
- $\Delta V = 0.36$ m/s.

States and control with $\epsilon = 10^{-6}$

Performances with $\epsilon = 10^{-6}$
1D fuel optimal lunar landing

• The 1D fuel optimal landing has been chosen to test the ability of the proposed framework to deal with possible discontinuities in the control.

• The fuel optimal problem is intrinsically constrained between a maximum and a minimum value. The use of a saturation function could result in a convenient choice.

\[
\min J = \int_{t_0}^{t_f} u(t) \, dt + \int_{t_0}^{t_f} \epsilon w^2(t) \, dt \quad \text{subject to:} \begin{cases}
\dot{h} = v \\
\dot{v} = -g + u \\
t_0 \leq t \leq t_f \\
h(t_0) = h_0 \\
v(t_0) = v_0 \\
h(t_f) = h_f \\
v(t_f) = v_f \\
u_{\text{min}} \leq u \leq u_{\text{max}}
\end{cases}
\]

• The Hamiltonian of the problem is:

\[
H = u + \lambda_h v + \lambda_v (-g + u) + \epsilon w^2 + \nu (u - \phi(w))
\]
1D fuel optimal lunar landing

- By using the CEs, the latent solutions are approximated.

\[
\begin{align*}
    h &= \left(\sigma - \Omega_1 \sigma_0 - \Omega_2 \sigma_f - \Omega_3 \sigma'_0 - \Omega_4 \sigma'_f\right)^T \beta_h + \Omega_1 h_0 + \Omega_2 h_f + \frac{\Omega_3 v_0}{b^2} + \frac{\Omega_4 v_f}{b^2} \\
    \lambda_h &= \sigma^T \beta_{\lambda_h} \\
    \lambda_v &= \sigma^T \beta_{\lambda_v} \\
    w &= \sigma^T \beta_w \\
    \nu &= \sigma^T \beta_{\nu}
\end{align*}
\]

- The first order necessary conditions and the corresponding losses are:

\[
\begin{align*}
    \dot{h} &= \frac{\partial H}{\partial \lambda_h} = v \\
    \dot{v} &= \frac{\partial H}{\partial \lambda_v} = -g + u \\
    \dot{\lambda}_h &= -\frac{\partial H}{\partial h} = 0 \\
    \dot{\lambda}_v &= -\frac{\partial H}{\partial \nu} = -\lambda_h \\

    \frac{\partial H}{\partial u} &= 1 + \lambda_v + \nu = 0 \\
    \frac{\partial H}{\partial w} &= 2\epsilon w - \nu \phi'(w) = 0 \\
    \frac{\partial H}{\partial \nu} &= u - \phi(w) = 0
\end{align*}
\]

\[
\begin{align*}
    \mathcal{L}_s &= \dot{v} + g - \phi(w) \\
    \mathcal{L}_{\lambda_h} &= \dot{\lambda}_h \\
    \mathcal{L}_{\lambda_v} &= \dot{\lambda}_v + \lambda_h \\
    \mathcal{L}_u &= 1 + \lambda_v + \nu \\
    \mathcal{L}_w &= 2\epsilon w - \nu \phi'(w)
\end{align*}
\]
1D fuel optimal lunar landing

- The following conditions are imposed: $r_0 = 10 \text{ m}$, $v_0 = -2 \text{ m/s}$, $t_f = 4.2 \text{ s}$, $g = 1.62 \text{ m/s}^2$.
- A continuation procedure has been applied to decrease both the value of $\epsilon$ from $100$ to $10^{-15}$ and the coefficient $c$ of the saturation function from $c_0$ to $c_f$.
- The parameters employed for the simulation are:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$L$</th>
<th>$\sigma$</th>
<th>$[u_{\min}, u_{\max}]$ m/s$^2$</th>
<th>$c_0$</th>
<th>$c_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>30</td>
<td>Logistic</td>
<td>[0.3]</td>
<td>100</td>
<td>10</td>
</tr>
</tbody>
</table>

States, control and performances with $\epsilon = 10^{-15}$
Conclusions

• A new PINN-based algorithm is proposed to solve optimal control problems with control constraints using saturation functions.

• X-TFC has been successfully employed to solve the TPBVP arising from the application of the PMP to the Hamiltonian of the problem.

• COCPs for space guidance have been solved
  • Minimum time-energy optimal Halo-Halo transfer.
  • 1D fuel optimal lunar landing. The bang-bang and bang-off-bang type of solution could be well approximated by means of saturation functions.

• The proposed algorithm can be potentially suitable for real time applications if a compromise between the optimality of the results and the precision of the dynamics is carried out.

➢ The possibility to solve more complex fuel optimal problems will be taken into account considering a combination of saturation functions.
THANK YOU FOR YOUR ATTENTION!

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